MATH 2040 Lecture 7 (Sep 29, 2016)

1^{st} sufficency test: $A \in M_{n\times n}(\mathbb{F})$ has n distinct e.values in $\mathbb{F}$

$\mathbb{R}/\mathbb{C}$ $\Rightarrow$ A diagonalizable.

2^{nd} sufficency test: $A \in M_{n\times n}(\mathbb{R})$ symmetric (i.e. $A^T = A$)

$\mathbb{R}$ $\Rightarrow$ A diagonalizable.

Necessary & sufficient condition ($\mathbb{R}/\mathbb{C}$)

$A \in M_{n\times n}(\mathbb{F})$ is diagonalizable

iff (i) $f(t)$ "splits over $\mathbb{F}$", i.e.

$$f(t) = (-1)^n (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$$

for some $\lambda_i \in \mathbb{F}$, m_i = (algebraic) multiplicity of λ_i

(ii) $\dim E_{\lambda_i} = m_i$ for each i.

E.g. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ NOT diagonalizable.

$\lambda_1 = 1$ $\quad f(t) = (t-1)^2$ $\quad m_1 = 2$

$\dim E_1 = 1 < 2 = m_1$

E.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$f(t) = t^2+1$ does not split over $\mathbb{R}$ $\Rightarrow$ NOT diag. / $\mathbb{R}$

$= (t+i)(t-i)$ splits over $\mathbb{C}$ $\Rightarrow$ diag. / $\mathbb{C}$

1^{st} suff. test

Necessity: Assume $A \in M_{n\times n}(\mathbb{F})$ diag. $\quad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

($\Rightarrow$) By def$^{\underline{n}}$, $\exists\, Q$ invertible s.t.

$$Q^{-1} A Q = D = \begin{pmatrix} \overset{m_1}{\boxed{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}}} & & & \\ & \overset{m_2}{\boxed{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}}} & & \\ & & \boxed{\ddots} & \\ & & & \overset{m_k}{\boxed{\begin{matrix}\lambda_k & & \\ & \ddots & \\ & & \lambda_k\end{matrix}}} \end{pmatrix}$$

$A \sim D \Rightarrow$ (1) same eigenvalues

(2) "same" eigenvectors $(\vec{v} \longleftrightarrow Q\vec{v})$

Char. poly of D (of A) $= f(t) = \det(D - tI)$

$$= (\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2}\cdots(\lambda_k - t)^{m_k}$$

$$= \underbrace{(-1)^n (t-\lambda_1)^{m_1}\cdots(t-\lambda_k)^{m_k}}_{\text{splits}} \Rightarrow \text{(i)}$$

$$E_{\lambda_i} = N(D - \lambda_i I)$$

of D

$$= N\begin{pmatrix} \boxed{\lambda_1 - \lambda_i} & & \\ & \overset{m_i}{\boxed{0}} & \\ & & \boxed{\lambda_k - \lambda_i} \end{pmatrix}$$

$\dim = m_i \Rightarrow$ (ii)

$\square$

Example:

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}, \text{ is it diagonalizable } / \mathbb{R} \text{ ?}$$

not symmetric

Solution:

$$f(t) = \det(A - tI)$$

$$= \det \begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (4-t)\left[(3-t)(4-t)\right] + 1 \cdot \left[-(3-t)\right]$$

$$= (3-t)\left[(4-t)^2 - 1\right]$$

$$= -(t-3)\left[(4-t+1)(4-t-1)\right]$$

$$= -(t-3)(5-t)(3-t)$$

$$= -(t-3)^2(t-5) = 0 \Rightarrow \boxed{\begin{matrix} \lambda_1 = 3, & \lambda_2 = 5 \\ m_1 = 2, & m_2 = 1 \end{matrix}}$$

For $\lambda_1 = 3$,

$$E_3 = N(A - 3I) = N\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$\boxed{\dim E_3 = 2 = m_1}$$

For $\lambda_2 = 5$,

$$E_5 = N(A - 5I) = N\begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\}$$

$$\boxed{\dim E_5 = 1 = m_2}$$

$\Rightarrow$ A <u>is</u> diagonalizable!

Q: What is an eigenbasis?

$$\beta = \{\underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{E_{\lambda_1}}, \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{E_{\lambda_2}}\}$$ eigenbasis

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow Q^{-1} A Q = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 5 \end{pmatrix}$$